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# Deconstructing the D0-D6 system

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## Abstract

We find the complete classical moduli space of two-centered supersymmetric solutions carrying D0 and D6 brane charge in the STU model delimited by walls of marginal stability of co-dimension one. U-duality guarantees our conclusions hold for any BPS state with negative quartic invariant. The analysis explicitly shows that the conditions of marginal stability, *i.e.* the integrability conditions, are generically insufficient to provide a regular supergravity solution in this model.

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# 1 Introduction

String theory provides a microscopic description of black holes as bound states of D-branes and other solitonic objects. Significant progress has been achieved by understanding the structure of these bound states and how these features manifest in supergravity [1, 2, 3, 4, 5, 6, 7, 8, 9]. In asymptotically flat four dimensional spacetimes, some BPS states in string theory with a fixed set of charges can be described as a single center black hole and/or as a multi-centered solution [10, 11, 12, 13, 14]. Generically, the asymptotic conserved charges and a set of regularity conditions define a classical moduli space, which should be after proper quantization in agreement with the microscopic theory in appropriate regimes [15, 16, 17]. The split attractor flow conjecture [3, 18] proposes a description of this moduli space for half BPS states in  $\mathcal{N} = 2$  theories in 4D. The basic idea is that a solution will exist if there is an attractor flow tree in moduli space that terminates on the attractor points of the constituents charges. The bifurcation points of the tree correspond to the regions in moduli space where the state becomes marginally stable and breaks apart.

For a given bound state with fixed charge vector, *a priori* there may be an infinite number of ways to split up its charge into bound state composites. This would lead to an infinite degeneracy which is known not to occur. As conjectured by the split attractor flow, there should be physical requirements that will only allow a finite number of such decompositions. In supergravity, these translate to kinematic conditions (*e.g.* mass and charge conservation) and dynamical conditions (*e.g.* smooth geometry) on the multi-centered solution describing the bound state. Our motivation is to investigate these conditions in detail and classify all possible composites for a given total charge.

Answering this question is extremely difficult for a generic supersymmetric bound state. In this paper, we will focus on a particular class of BPS states with negative quartic invariant,  $\Delta$ , and study the realization of supersymmetric states in the STU model. States with  $\Delta < 0$  are particularly interesting because they will always correspond to polar states in the BPS branch, *i.e.* the supergravity description is always multi-centered. In addition, U-duality guarantees that we can choose a U-dual frame where the system only carries D0 and D6-brane charges [19]. This is an extremely simple charge vector which will allow us to explicitly construct the bound states in a fairly straight forward manner.

The D0-D6 system was analyzed in [20, 21, 22, 23, 24], where the existence of supersymmetric bound state was guaranteed if a sufficiently large  $B$ -field was turned on. This condition defines a region of moduli space where the state exists, and it is delimited by a wall of marginal stability of co-dimension one. More recently, these bound states have been described in the large volume approximation as two-centered supergravity configurations [18], where one center carries D0-charge and the second one carries D6-charge. The location in moduli space where the bound state starts to exist in the classical theory coincides with the wall of marginal stability derived in the weakly coupled description of the D-brane system.

In the supergravity approximation, it is natural to ask whether there are any other supersymmetric two-centered regular configurations carrying the same charge as a D0-D6

bound state, but with different constituent charges.<sup>1</sup> In the following we will determine all such configurations that are bounded by co-dimension one walls of marginal stability in moduli space. In principle one could consider solutions with more than two centers, but the integrability conditions will generate walls of higher co-dimension.

Our strategy will consist of two main steps: an algebraic classification of the potential composites of the bound state and the supergravity description of the latter. In the first step, we will determine all possible candidate constituents building a D0-D6 bound state, consistent with supersymmetry, and conservation of mass and charge. Knowing the composite charges and fixing the moduli at infinity, we can compute the central charges (in the large volume limit) associated with these states and study the regions in moduli space where they remain finite. Furthermore, we can also determine the loci in moduli space where walls of marginal stability exist.

In the second step, we will find the supergravity realization for these bound states as two-centered configurations and study their regularity. We fix both the charges at infinity and at each center (using the results in the first part of our analysis), and determine the distance scale between the centers by solving the integrability condition. This is guaranteed to be positive in the same region defined by the wall of marginal stability, but it is not enough to assure the regularity of the supergravity configuration. This requires, in addition, the positivity of an scalar function  $\Sigma^2$  and the absence of closed timelike curves (CTCs). We will explicitly see that these requirements are non-trivial. In particular, we will prove that all the conditions required on the central charges in the first part of our analysis are necessary, but still not sufficient to guarantee the existence of the bound state in supergravity.

One main lesson of our analysis is to explicitly show that the kinematic conditions derived from supersymmetry, in addition to having well-defined composite states, are not enough to assure the stability of the bound state. There are some non-trivial dynamical conditions which in supergravity arise from requiring a regular geometry. It would be interesting to understand how these conditions are translated on the microscopic Hilbert space of BPS states.

This paper is organized as follows. In section 2, we start by briefly reviewing the STU model and its most general stationary BPS solutions. We comment on the connection between the zeroes in the central charge and the location of the walls of marginal stability. We also review how U-duality orbits allow us to focus on the D0-D6 system. In section 3, we first determine all 1/4 and 1/2 BPS charge vectors consistent with conservation of mass and charge. We analyze the conditions under which their central charges do not vanish and determine the equations describing the walls of marginal stability in each case. In section 4, we study the regularity of the corresponding two-centered supergravity configurations. In section 5, we extend our analysis to include 1/8 constituent BPS states and we finish with some conclusions.

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<sup>1</sup>In some recent papers [25, 26, 27] similar questions have been discussed for both the BPS and non-BPS branch of the D0-D6 system.

## 2 D0-D6 in the STU model

### 2.1 STU model

We begin the discussion with a brief overview of four dimensional BPS configurations in supergravity. Our focus is on the  $\mathcal{N} = 2$  theory known as the STU-model [28, 29, 30]. We will interpret the model in terms of type IIA string theory compactified on a  $T^6$  of the form  $T^2 \times T^2 \times T^2$ . The D0/D2/D4/D6-branes wrapping the various cycles of  $T^6$  give rise to four magnetic and four electric charges that are assembled into the charge vector

$$\Gamma = (p^0, p^A; q_A, q_0) , \quad (2.1)$$

with  $A = 1, 2, 3$ , and each component representing (D6, D4, D2, D0) brane charges respectively.  $\mathcal{N} = 2$  theories are characterized by a prepotential  $F$ . In the STU model the prepotential and its derivatives are

$$F = -\frac{X^1 X^2 X^3}{X^0} , \quad F_\Sigma = \frac{\partial F}{\partial X^\Sigma} . \quad (2.2)$$

We gauge fix the projective coordinates  $X^\Lambda$  ( $\Lambda = 0, 1, 2, 3$ ) so that  $X^0 = 1$ , and define  $X^A \equiv z^A = B^A + iJ^A$ . Then the Kähler potential is given by

$$K = -\ln i(F_\Sigma \bar{X}^\Sigma - \bar{F}_\Sigma X^\Sigma) = -\ln(8J^1 J^2 J^3) , \quad (2.3)$$

where  $X^\Lambda \bar{F}_\Lambda = -X^0 \bar{F}_0 + X^A \bar{F}_A$ , and the central charge reads

$$Z = e^{K/2} [X^\Lambda q_\Lambda - F_\Lambda p^\Lambda] = e^{K/2} [p^0 z^1 z^2 z^3 - \frac{1}{2} s_{ABC} p^A z^B z^C + z^A q_A - q_0] , \quad (2.4)$$

where the only non-vanishing intersection numbers are  $s_{123} = 1$  and cyclic permutations.

#### 2.1.1 BPS solutions

The most general stationary but non-static BPS configurations solving the STU equations of motion were constructed in [11, 12, 13] and are reviewed in appendix A. Their metrics

$$ds^2 = -\frac{1}{\Sigma}(dt + \omega)^2 + \Sigma ds_{\mathbb{R}^3}^2 , \quad (2.5)$$

are described by the one-form  $\omega$  defined on  $\mathbb{R}^3$  and the scalar function  $\Sigma^2$

$$\begin{aligned} \Sigma^2(H) = & - (H_\Lambda H^\Lambda)^2 + 4 (H^1 H_1 H^2 H_2 + H^1 H_1 H^3 H_3 + H^2 H_2 H^3 H_3) \\ & - 4H^0 H_1 H_2 H_3 - 4H_0 H^1 H^2 H^3 , \end{aligned} \quad (2.6)$$

depending on eight harmonic functions  $(H^\Lambda, H_\Lambda)$

$$H^\Lambda = \sum_{i=1}^N \frac{p_i^\Lambda}{|\vec{x} - \vec{x}_i|} + h^\Lambda , \quad H_\Lambda = \sum_{i=1}^N \frac{q_\Lambda^i}{|\vec{x} - \vec{x}_i|} + h_\Lambda . \quad (2.7)$$

These harmonic functions encode all the information about the conserved charges and moduli. The total charge  $\Gamma = (p^\Lambda; q_\Lambda)$  is split into  $N$  centers, each carrying charge vector  $\Gamma_i = (p_i^\Lambda; q_\Lambda^i)$  so that  $p^\Lambda = \sum_i p_i^\Lambda$  and  $q_\Lambda = \sum_i q_\Lambda^i$ . The moduli values at infinity ( $z_\infty^A$ ) and the charge vector  $\Gamma$  define a total central charge  $Z = |Z| e^{i\alpha}$ . These determine the set of constants  $h = (h^\Lambda; h_\Lambda)$  (see (A.3)) by requiring the metric to be asymptotically flat and to solve the integrability conditions below.

Such solutions are regular if they satisfy:

1. integrability conditions which guarantee the absence of Dirac-Misner strings

$$\sum_{b \neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} = \langle h, \Gamma_a \rangle, \quad \text{with} \quad \langle \Gamma_i, \Gamma_j \rangle = -p_i^0 q_0^j + p_i^A q_A^j - q_{iA}^j p_j^A + q_0^i p_j^0, \quad (2.8)$$

2. positivity of the function  $\Sigma^2$ , *i.e.*  $\Sigma^2 > 0 \ \forall \vec{x} \in \mathbb{R}^3$ ,
3. absence of CTCs, *i.e.*  $\Sigma^2 - \omega_i \omega^i > 0 \ \forall \vec{x} \in \mathbb{R}^3$ , and absence of singularities in the moduli fields.

Close to each pole  $\vec{x}_i$ , the attractor equations govern the behavior of the function  $\Sigma^2$  and fixes the scalar moduli [31, 32, 33]. In particular, the leading term as  $\vec{x} \rightarrow \vec{x}_i$  is

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_i) = \frac{\Delta_i}{|\vec{x} - \vec{x}_i|^4} + \mathcal{O}(|\vec{x} - \vec{x}_i|^{-3}), \quad (2.9)$$

where  $\Delta_i$  is the quartic invariant associated to the charge vector  $\Gamma_i$ . In the STU model, the quartic invariant of the U-duality group  $(\text{SL}(2, \mathbb{R}))^3$  is given by

$$\Delta = -(p^\Lambda q_\Lambda)^2 + 4(p^1 q_1 p^2 q_2 + p^1 q_1 p^3 q_3 + p^2 q_2 p^3 q_3) - 4p^0 q_1 q_2 q_3 - 4q_0 p^1 p^2 p^3 \quad (2.10)$$

with

$$p^\Lambda q_\Lambda \equiv -p^0 q_0 + p^1 q_1 + p^2 q_2 + p^3 q_3.$$

The value of  $\Delta$  determines the amount of supersymmetry preserved by the system [34]. For  $\Delta > 0$  we have a BPS black hole preserving 1/8 supercharges; single centered solutions with  $\Delta < 0$  are non-BPS; and if  $\Delta = 0$  the system can preserve 1/8 or more supercharges. As reviewed in (A.11) different BPS states have different scaling in  $|\vec{x} - \vec{x}_i|$  [35].

**Two-centered solutions.** The bound states we will construct in the later sections consist on only two centers, hence it will be useful to simplify the above expressions for such case. We will use a similar notation to the one discussed in [25]. For any two-centered configuration, we can always take the first center at the origin and the second on the  $z$ -axis at distance  $R$ , carrying generic charge vectors

$$\Gamma_1 = (p_1^\Lambda, q_\Lambda^1), \quad \vec{x}_1 = (0, 0, 0), \quad (2.11)$$

$$\Gamma_2 = (p_2^\Lambda, q_\Lambda^2), \quad \vec{x}_2 = (0, 0, R), \quad (2.12)$$

with  $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ . The harmonic functions are given by (2.7), and by using standard spherical coordinates on  $\mathbb{R}^3$  their radial dependence simplifies to

$$|\vec{x} - \vec{x}_1|^2 = r^2, \quad \Theta^2 \equiv |\vec{x} - \vec{x}_2|^2 = r^2 - 2rR \cos \theta + R^2. \quad (2.13)$$

The integrability conditions (2.8) are

$$\begin{aligned} \frac{\langle \Gamma_1, \Gamma_2 \rangle}{R} &= \frac{\text{Im}(Z_1 \bar{Z}_2)}{2|Z_{1+2}|} \\ &= h^\Lambda q_\Lambda^1 - h_\Lambda p_1^\Lambda = -h^\Lambda q_\Lambda^2 + h_\Lambda p_2^\Lambda. \end{aligned} \quad (2.14)$$

Next, the one-form is determined by integrating (A.5). Using (2.14), the right hand side of (A.5) reads

$$\langle dH, H \rangle = -\frac{\langle \Gamma_1, \Gamma_2 \rangle}{R} (dr^{-1} - d\theta^{-1}) + \langle \Gamma_1, \Gamma_2 \rangle (\Theta^{-1} dr^{-1} - r^{-1} d\theta^{-1}) \quad (2.15)$$

Integrating the above expression, we obtain

$$\omega = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{R} \left[ 1 - \frac{r+R}{\Theta} \right] (1 - \cos \theta) d\phi, \quad (2.16)$$

where we fixed the integration constant so that our solutions are asymptotically flat, *i.e.*  $\omega \rightarrow 0$  at infinity, and it avoids Dirac-Misner singularities at  $\theta = 0, \pi$ . Knowing  $\Sigma^2$  and the one-form  $\omega$ , the sufficient condition to ensure the absence of CTCs is

$$\Sigma^2 r^2 \sin^2 \theta > (\omega_\phi)^2. \quad (2.17)$$

## 2.2 D0-D6 bound states

We want to identify the possible different representations of the D0-D6 system as a BPS bound state in the STU model. In the notation introduced above, D0-D6 corresponds to turning only  $p^0$  and  $q_0$  in (2.1). The quartic invariant (2.10) is then given by  $\Delta = -(p^0 q_0)^2$ . Since the value of the  $\Delta$  is negative, it is clear that  $\Sigma^2$  is not positive definite. In particular, close to the charge source location ( $\vec{x} \rightarrow 0$ ),

$$\Sigma^2 \rightarrow -\left(\frac{p^0 q_0}{|\vec{x}|^2}\right)^2 + \mathcal{O}(|\vec{x}|^{-3}).$$

This observation is consistent with the existence of loci in moduli space where the total D0-D6 central charge vanishes

$$Z_{D0-D6} = e^{K/2} (p^0 z^1 z^2 z^3 - q_0) = 0 \quad \Leftrightarrow \quad \text{Im}(z^1 z^2 z^3) = 0, \quad \text{Re}(z^1 z^2 z^3) = \frac{q_0}{p^0}$$

Whenever this occurs at a finite point in moduli space, the BPS state does not exist, as argued in [36, 37, 38, 39]. We will use this criterion all along this work.



The above conclusion was reached in the supergravity approximation and assuming the realization of the state in terms of a single center configuration. But D0-D6 states may allow different descriptions as a function of the string coupling constant. The problem of adhering D0-branes to D6-branes in a supersymmetric manner was studied in [23].<sup>2</sup> It was found that a supersymmetric branch exists for sufficiently large  $B$ -fields such that

$$\frac{1}{2}s_{ABC}B^A J^B B^C \geq J^1 J^2 J^3 . \quad (2.18)$$

In recent work in the supergravity literature [11], these supersymmetric bound states were identified with two-centered supergravity configurations carrying D6-brane and D0-brane charges at each center. These are characterised by two charge vectors

$$\Gamma_1 = (p^0, \vec{0}; \vec{0}, 0) \quad \text{and} \quad \Gamma_2 = (0, \vec{0}; \vec{0}, q_0) ,$$

sourced at points  $\vec{x}_1$  and  $\vec{x}_2$  separated by a distance  $R = |\vec{x}_1 - \vec{x}_2|$ , which is uniquely determined by solving the integrability condition

$$R = \frac{\langle \Gamma_1, \Gamma_2 \rangle |Z_{1+2}|}{2 \operatorname{Im}(Z_1 \bar{Z}_2)} . \quad (2.19)$$

The separation becomes infinite precisely when the equality in (2.18) is saturated, which corresponds to the location of a wall of marginal stability

$$\operatorname{Im}(Z_1 \bar{Z}_2) = 0 .$$

This is interpreted as the disappearance of the bound state when crossing such wall. Thus, for the bound state to exist the separation scale  $R$  must be physical, *i.e.*  $\langle \Gamma_1, \Gamma_2 \rangle \operatorname{Im}(Z_1 \bar{Z}_2) > 0$  is a necessary condition. This can be confirmed by computing the number of BPS states as a function of the moduli and proving the existence of a jump in the mathematical index that accounts for these degeneracies [18, 14, 7].<sup>3</sup>

### 2.3 Walls of marginal stability: systematics

Given the connection between the existence of a D0-D6 BPS bound state and a two-centered supergravity configuration, it is natural to wonder whether there could be other two-centered configurations with the same charges at infinity but different charge split decomposition, *i.e.* different pole charge vectors  $\{\Gamma_1, \Gamma_2\}$ .<sup>4</sup> This requires us to identify the different walls of marginal stability where the split may occur. Given a BPS state with charge vector  $\Gamma_{1+2}$ , mass  $M_{1+2} = |Z_{1+2}|$  and central charge  $Z_{1+2} = e^{i\alpha}|Z_{1+2}|$ , the necessary conditions that define a wall of marginal stability are

$$\Gamma_{1+2} = \Gamma_1 + \Gamma_2 \quad (2.20a)$$

$$|Z_{1+2}| = |Z_1| + |Z_2| \quad (2.20b)$$

---

<sup>2</sup>See also [22, 21, 20, 24]

<sup>3</sup>The agreement of the BPS moduli space for the supergravity solution and the open string perturbative analysis was explained in [11] and compared to its non-BPS branch in [27].

<sup>4</sup>In this work, we will focus on the STU truncation of the full  $\mathcal{N} = 8$  supergravity, and the reader should be aware that our conclusions may not apply to the full theory.

where  $\{Z_i, \Gamma_i\}$  with  $i = 1, 2$  stand for the data of the bound state constituents once the wall is crossed. These conditions assure conservation of charge and mass at the wall of marginal stability. They are equivalent to solving

$$\begin{aligned} \text{Im} \left( Z_1 \bar{Z}_2 \right) &= 0, \\ \text{Re} \left( Z_1 \bar{Z}_2 \right) &> 0. \end{aligned} \quad (2.21)$$

**D0-D6 walls.** Given a D0-D6 system where

$$\Gamma_{\text{D0-D6}} = (p^0, \vec{0}; \vec{0}, q_0),$$

its most general split into two vectors  $\Gamma_1$  and  $\Gamma_2$ , consistent with charge conservation, is

$$\begin{aligned} \Gamma_1 &= (P^0, P^A; Q_A, Q_0), \\ \Gamma_2 &= (p^0 - P^0, -P^A; -Q_A, q_0 - Q_0). \end{aligned}$$

The central charges of all the above charge vectors are

$$Z_{\text{D0-D6}} = e^{K/2} (p^0 z^1 z^2 z^3 - q_0) \equiv e^{K/2} Y_{\text{D0-D6}}, \quad (2.22)$$

and

$$Z_1 = e^{K/2} Y_1, \quad Z_2 = e^{K/2} (Y_{\text{D0-D6}} - Y_1), \quad (2.23)$$

where we defined

$$Y_1 \equiv \left( P^0 z^1 z^2 z^3 - \frac{1}{2} s_{ABC} P^A z^B z^C + z^A Q_A - Q_0 \right). \quad (2.24)$$

Let us analyze the consequences due to the existence of a wall of marginal stability on our general split described above.<sup>5</sup> From the condition (2.21) :

$$Z_1 \bar{Z}_2 = \bar{Z}_1 Z_2 \Leftrightarrow \frac{Z_1}{\bar{Z}_1} = \frac{Z_2}{\bar{Z}_2} \Leftrightarrow \frac{Y_1}{\bar{Y}_1} = \frac{Y_{\text{D0-D6}} - Y_1}{\bar{Y}_{\text{D0-D6}} - \bar{Y}_1} \Leftrightarrow \frac{Y_1}{\bar{Y}_1} = \frac{Y_{\text{D0-D6}}}{\bar{Y}_{\text{D0-D6}}} \quad (2.25)$$

Notice  $\alpha_1 = \alpha_2 + n\pi$ , but also  $\alpha_1 = \alpha_{\text{D0-D6}} + m\pi$  for  $n, m \in \mathbb{Z}$ . In other words, this condition still allows both aligned and misaligned central charges. Also the last equality would be perfectly consistent with an split of the form  $\Gamma_2 \rightarrow \Gamma_1 + \Gamma_{\text{D0-D6}}$ , which is not what we are interested in studying.

It is the second condition in (2.21)

$$\text{Re} (Z_1 \bar{Z}_2) = |Z_1| |Z_2| \cos(\alpha_1 - \alpha_2) > 0,$$

that guarantees both split charges are aligned. Furthermore, since  $Z_2 = Z_{\text{D0-D6}} - Z_1$ , it follows

$$|Z_2| e^{i\alpha_1} = ((-1)^m |Z_{\text{D0-D6}}| - |Z_1|) e^{i\alpha_1}, \quad (2.26)$$

which is only consistent when all three charges involved in the split are aligned. Thus, it is the second condition in (2.21) that breaks the reversibility of the split, disallowing channels such as  $\Gamma_2 \rightarrow \Gamma_1 + \Gamma_{\text{D0-D6}}$ , since

$$M_{\text{D0-D6}} = |Z_{\text{D0-D6}}| = M_1 + M_2 = |Z_1| + |Z_2|.$$

---

<sup>5</sup>At this point, we assume that all charge vectors are supersymmetric. We will study the requirements later.

## 2.4 U-duality orbits

The D0-D6 system is a particular example of a state with negative quartic invariant ( $\Delta < 0$ ). Any other such state would be subject to the same considerations discussed so far. Thus, it is important to determine whether there exists any U-duality transformation relating these different states so that the conclusions reached for the D0-D6 can be extended to the full subclass of these states. It was proved in [19] that all states with  $\Delta < 0$  belong to the same U-duality orbit. This is shown recalling that charges in the STU model transform in the  $(2, 2, 2)$  representation of the  $(\text{SL}(2, \mathbb{R}))^3$  duality symmetry group. For completeness, we include their proof below.

Let us parameterize the three  $\text{SL}(2, \mathbb{R})$  matrices building the U-duality group  $(\text{SL}(2, \mathbb{R}))^3$  as

$$M_A = \begin{pmatrix} a^A & b^A \\ c^A & d^A \end{pmatrix} \quad \text{with} \quad \det(M_A) = 1, \quad A = 1, 2, 3. \quad (2.27)$$

Consider a charge vector with arbitrary charges  $(P^0, P^A; Q_A, Q_0)$  and the vector  $(p^0, \vec{0}; \vec{0}, q_0)$ . Given the transformation properties of the charges, these two set of charges are related by the set of constraints

$$-Q_0 = a^1 a^2 a^3 q_0 + b^1 b^2 b^3 p^0, \quad (2.28)$$

$$Q_A = -\frac{1}{2} s_{BCD} c^B a^C a^D q_0 + \frac{1}{2} s_{BCD} d^B b^C b^D p^0, \quad (2.29)$$

$$P^A = -\frac{1}{2} s_{BCD} a^B c^C c^D q_0 + \frac{1}{2} s_{BCD} b^B d^C d^D p^0, \quad (2.30)$$

$$P^0 = c^1 c^2 c^3 q_0 + d^1 d^2 d^3 p^0. \quad (2.31)$$

This system is solved by the following set of matrices [19] :

$$M_A = -\frac{\text{sgn}(\xi)}{\sqrt{(\psi_A + \rho_A)} \xi} \begin{pmatrix} \psi_A \xi & -\rho_A \\ \xi & 1 \end{pmatrix} \Leftrightarrow M_A^{-1} = -\frac{\text{sgn}(\xi)}{\sqrt{(\psi_A + \rho_A)} \xi} \begin{pmatrix} 1 & \rho_A \\ -\xi & \psi_A \xi \end{pmatrix},$$

with

$$\xi = \left( \frac{p^0}{q_0} \right)^{1/3} \left( \frac{2P^1 P^2 P^3 + P^0 (\sqrt{-\Delta} - P^\Lambda Q_\Lambda)}{2P^1 P^2 P^3 - P^0 (\sqrt{-\Delta} - P^\Lambda Q_\Lambda)} \right)^{1/3} \in \mathbb{R}, \quad (2.32)$$

$$\psi_A = \frac{\sqrt{-\Delta} + P^\Lambda Q_\Lambda - 2P^A Q_A}{s_{ABC} P^B P^C - 2P^0 Q_A} \in \mathbb{R} \text{ (no sum on A)}, \quad (2.33)$$

$$\rho_A = \frac{\sqrt{-\Delta} - P^\Lambda Q_\Lambda + 2P^A Q_A}{s_{ABC} P^B P^C - 2P^0 Q_A} \in \mathbb{R} \text{ (no sum on A)}. \quad (2.34)$$

These transformations preserve the value of  $\Delta$ , *i.e.*

$$\Delta = -(p^0 q_0)^2 = -4Q_0 P^1 P^2 P^3 - 4P^0 Q_1 Q_2 Q_3 - (P^\Lambda Q_\Lambda)^2 + 4 \sum_{A < B} P^A Q_A P^B Q_B.$$

Thus, all the states in the orbit have negative quartic invariant.

As emphasized in [19], the above matrices are not the most general ones that can be constructed connecting states with negative quartic invariant. One could introduce a triple  $\xi_A$  satisfying the constraint  $\xi_1 \xi_2 \xi_3 = \xi^3$ , a feature that was already alluded to in the context of extremal non-BPS black holes in [40].

This result guarantees that given a wall of marginal stability and a pair of bound state constituents in the D0-D6 frame, they also exist in any other frame related to the latter.

### 3 Bound states of 1/4 and 1/2 BPS states

In this section we will determine the pairs of 1/4 and 1/2 BPS constituents that may form a bound state carrying only D0 and D6 brane charges by imposing local conditions on the system.<sup>6</sup> Our procedure is as follows: first, we solve for all composite vectors  $\Gamma_{1,2}$  that preserve at least 1/4 supercharges consistent with charge conservation; second, we analyze whether the states associated with such charge vectors exist; and finally, we derive the explicit equations for the walls of marginal stability.

#### 3.1 Classification of final states

Given a total charge vector  $\Gamma_{\text{D0-D6}} = (p^0, \vec{0}; \vec{0}, q_0)$ , we are looking for pairs of 1/4 and/or 1/2 BPS charge vectors  $\{\Gamma_1, \Gamma_2\}$  such that

$$\Gamma_{\text{D0-D6}} = \Gamma_1 + \Gamma_2 , \quad (3.1)$$

and with quartic invariant  $\Delta$  satisfying [41]

$$\Delta = 0 \quad \text{and} \quad \frac{\partial \Delta}{\partial q_\Lambda} = 0 , \quad \frac{\partial \Delta}{\partial p^\Lambda} = 0 . \quad (3.2)$$

In appendix B we present a detailed derivation for the general solution to these equations. There we argue that any charge vector satisfying (3.2) can be written as

$$(\beta_1 P^0, \beta_2 P^0, \alpha_1 P^2, \alpha_1 P^3; \beta_1 Q_0, \alpha_2 P^3, \alpha_2 P^2, \beta_2 Q_0) \quad (3.3)$$

with  $\alpha_{1,2}$  and  $\beta_{1,2}$  constants. Imposing the conditions (3.2) on (3.3) reduces to

$$\begin{aligned} P^0 P^2 P^3 \alpha_{1,2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) &= 0 , \\ Q_0 P^2 P^3 \alpha_{1,2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) &= 0 , \\ Q_0 P^0 P^{2,3} \beta_{1,2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) &= 0 . \end{aligned} \quad (3.4)$$

Consider two such charge vectors  $\{\Gamma_1, \Gamma_2\}$  consistent with charge conservation (3.1) :

$$\Gamma_1 = (-abp^0, -bcp^0, \alpha_1 p^2, \alpha_2 p^3; adq_0, \alpha_2 p^3, \alpha_2 p^2, cdq_0) \quad (3.5)$$

$$\Gamma_2 = (cdp^0, bcp^0, -\alpha_1 p^2, -\alpha_2 p^3; -adq_0, -\alpha_2 p^3, -\alpha_2 p^2, -abq_0) \quad (3.6)$$

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<sup>6</sup>We will discuss the possibility of 1/8 BPS constituents in a later section. One could also consider n-state splits, but these are necessarily co-dimension larger than one.

with

$$cd - ab = 1 . \quad (3.7)$$

Both charge vectors  $\Gamma_{1,2}$  must satisfy (3.4). Since  $p^0, q_0 \neq 0$ , conditions (3.4) on  $\Gamma_1$  reduce to

$$\begin{aligned} p^2 p^3 (a\alpha_2 - c\alpha_1) \alpha_{1,2} b &= 0 , & p^{2,3} (a\alpha_2 - c\alpha_1) abd &= 0 , \\ p^2 p^3 (a\alpha_2 - c\alpha_1) \alpha_{1,2} d &= 0 , & p^{2,3} (a\alpha_2 - c\alpha_1) bcd &= 0 , \end{aligned} \quad (3.8)$$

whereas for  $\Gamma_2$  we have

$$\begin{aligned} p^2 p^3 (d\alpha_2 - b\alpha_1) \alpha_{1,2} c &= 0 , & p^{2,3} (d\alpha_2 - b\alpha_1) acd &= 0 , \\ p^2 p^3 (d\alpha_2 - b\alpha_1) \alpha_{1,2} a &= 0 , & p^{2,3} (d\alpha_2 - b\alpha_1) abc &= 0 . \end{aligned} \quad (3.9)$$

There are three ways to simultaneously solve (3.8) and (3.9)

- i)*  $(a\alpha_2 - c\alpha_1) = (d\alpha_2 - b\alpha_1) = 0$  ,
- ii)*  $(a\alpha_2 - c\alpha_1) = 0$  &  $(d\alpha_2 - b\alpha_1) \neq 0$  ;  $(a\alpha_2 - c\alpha_1) \neq 0$  &  $(d\alpha_2 - b\alpha_1) = 0$  ,
- iii)*  $(a\alpha_2 - c\alpha_1) \neq 0$  &  $(d\alpha_2 - b\alpha_1) \neq 0$  .

For arbitrary values of  $\{a, b, c, d\}$  satisfying (3.7), condition *i)* is only solved if  $\alpha_1 = \alpha_2 = 0$ . These states, that we will refer to as **type I** states are

$$\Gamma_1^{(I)} = (-abp^0, [(-bcp^0)^A]; [(adq_0)_A], cdq_0) , \quad (3.10a)$$

$$\Gamma_2^{(I)} = (cdp^0, [(bcp^0)^A]; [(-adq_0)_A], -abq_0) . \quad (3.10b)$$

For conditions *ii)* and *iii)*, we find that the only non trivial solutions are obtained by setting either  $p^2$  (and/or  $p^3$ ) and one of the coefficients in (3.7) to zero. The resulting charge vectors, that we will refer to as **type II** states, are

$$\Gamma_1^{(II)} = (p^0, [p^A]; [q_B], 0) \quad (3.11a)$$

$$\Gamma_2^{(II)} = (0, [-p^A]; [-q_B], q_0) \quad (3.11b)$$

with  $A \neq B$ , and where the squared brackets are used to denote that there is a single charge of the vector  $p^A$  (or  $q_A$ ) turned on and the superscript (subscript) labels the component. For example,  $[p^1] = (p, 0, 0)$  and  $[q_2] = (0, q, 0)$ .

### 3.2 Existence of the split BPS states

Charge vectors (3.10) and (3.11) are supersymmetric, but this does not guarantee the state carrying them exists. Since we are eventually interested in interpreting two-centered supergravity configurations as bound states composed of the states associated with each center, we must first analyze when the individual states exist. This is a difficult question, specially for states with  $\Delta = 0$ , but one requirement we implement is that their central charges do not vanish. The conditions derived in this way match with the regularity of  $\Sigma^2$  in the single center supergravity realization of the given state.

**Type I states.** The central charges describing the charge vectors (3.10) are

$$Z_1^{(I)} = -e^{K/2} (b p^0 z^2 z^3 - d q_0) (a z^1 - c) , \quad (3.12a)$$

$$Z_2^{(I)} = e^{K/2} (c p^0 z^2 z^3 - a q_0) (d z^1 - b) , \quad (3.12b)$$

where we set  $A = 1$  in (3.10). Because of the factorized nature of these central charges, their zeroes can occur in either of their factors.

$Z_1^{(I)}$  can vanish when  $az_1 = c$ . This requires  $\text{Im} z_1 = 0$ , which is a singular point in moduli space, and lies beyond the regime of validity of our supergravity approximation. The second factor vanishes when  $bp^0 z^2 z^3 = dq_0$ , which is a complex equation. Assuming volumes never vanish, its imaginary part can be solved by

$$B^2 = -\frac{J^2}{J^3} B^3 .$$

Substituting this into its real part gives

$$-bp^0 \frac{J^2}{J^3} |z^3|^2 = dq_0 .$$

Notice that if any of the two parameters  $\{b, d\}$  vanish, the central charge will never vanish at a non-singular point in moduli space. This corresponds to the particular cases of D6-D4 ( $d = 0$ ) and D0-D2 ( $b = 0$ ). When  $d, b \neq 0$ , using the positivity of the volumes and  $p^0, q_0 \neq 0$ , we conclude:

1.  $Z_1^{(I)}$  has zeroes at non-singular points in moduli space *if*  $bd p^0 q_0 < 0$ .
2.  $Z_1^{(I)}$  has *no* zeroes at regular points in moduli space *if*  $bd p^0 q_0 > 0$ .
3.  $Z_1^{(I)}$  has *no* zeroes at regular points in moduli space whenever  $d = 0$  or  $b = 0$ .

$Z_2^{(I)}$  has an analogous structure to the one for  $Z_1^{(I)}$  and so are the conclusions:

1.  $Z_2^{(I)}$  has zeroes at non-singular points in moduli space *if*  $ac p^0 q_0 < 0$ .
2.  $Z_2^{(I)}$  has *no* zeroes at regular points in moduli space *if*  $ac p^0 q_0 > 0$ .
3.  $Z_2^{(I)}$  has *no* zeroes at regular points in moduli space whenever  $a = 0$  or  $c = 0$ .

From this analysis we conclude BPS constituents of type **I** will co-exist in the following cases:

1. If  $(a, b, c, d)$  are all non-vanishing, this requires

$$ac p^0 q_0 > 0 , \quad bd p^0 q_0 > 0 . \quad (3.13)$$

2. If  $d = 0$  ( $ab = -1$ ) and  $c \neq 0$ , this requires  $acp^0q_0 > 0$  or equivalently  $bc p^0q_0 < 0$ .<sup>7</sup>
3. When  $d = c = 0$  ( $ab = -1$ ) or  $a = b = 0$  ( $cd = 1$ ), the standard D0 + D6 split, constituents always exist.

It is interesting to relate these observations with the behavior of the supergravity solution near the charge source. All BPS states of type **I** have vanishing quartic invariant. The status of these states as supergravity solutions to the attractor equations is less obvious than those states having  $\Delta > 0$  due to the singular character of the solution at the pole (location of the charge). Generically one needs to include higher order corrections in the supergravity Lagrangian to properly describe these regions of spacetime. Despite this fact, one should still demand a smooth geometry at sufficiently large distance.

For example, BPS states carrying a single D-brane charge are well-defined states that preserve half of the supercharges. In particular their central charges never vanish on regular points of the moduli space. In the supergravity approximation, this translates into having  $\Sigma^2$  positive throughout space-time even though the size of the horizon is zero classically. For more general charge vectors with  $\Delta = 0$ , it is natural to analyze the behavior of the factor  $\Sigma^2$  as a function of the charges and moduli to determine the existence of the state. The attractor mechanism only fixes the value of  $\Sigma^2$  at the horizon to be proportional to  $\Delta$ . Thus, the dominant contribution to  $\Sigma^2$  very close to the charge source is no longer guaranteed to be independent of the moduli, and the positivity of  $\Sigma^2$  might not be satisfied.

Let us describe this more explicitly for generic type **I** states. Consider a single center BPS supergravity configuration realizing the state with central charge  $Z_1^{(I)}$  and charge vector  $\Gamma_1^{(I)}$ . The phase of the central charge satisfies :

$$|Z_1^{(I)}| \sin \alpha = e^{K/2} \left( -abp^0 \text{Im}(z^1 z^2 z^3) + cbp^0 \text{Im}(z^2 z^3) + adq_0 J^1 \right) .$$

For generic values of the parameters, the dominant contribution to  $\Sigma^2$  near the pole is given by

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_1) \rightarrow 16 e^K J^2 J^3 \frac{|az^1 - c|^2}{|\vec{x} - \vec{x}_1|^2} bdp^0 q_0 + \mathcal{O}(|\vec{x} - \vec{x}_1|^{-1}) \quad (3.14)$$

Notice that (3.14) diverges as  $1/r^2$ ,  $r$  being the distance to the pole, hence it corresponds to a 1/4 BPS state (see (A.11)). Furthermore,  $\Sigma^2$  is *only* positive when  $bdp^0 q_0 > 0$ , which matches the condition derived from requiring the absence of zeroes in  $Z_1^{(I)}$  at regular points in moduli space.

In the particular case  $d = 0$ , the dominant contribution to  $\Sigma^2$  in the same limit studied above is

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_1) \rightarrow |Z_1^{(I)}| \frac{4}{|\vec{x} - \vec{x}_1|} + \mathcal{O}(|\vec{x} - \vec{x}_1|^0) \quad (3.15)$$

The  $1/r$  divergence matches the 1/2 BPS character of this set of states, and (3.15) is *always* positive in this limit, in agreement with the regularity of the central charge for these states.

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<sup>7</sup>There is an analogous situation for  $b = 0$  ( $cd = 1$ ) and  $a \neq 0$ , which also requires  $acp^0q_0 > 0$ .

What we would like to emphasize is that the precise value of the dominant contribution to  $\Sigma^2$  does depend on the moduli turned on at infinity. In particular, it will generically depend on the total central charge phase  $\alpha$ .

**Type II split.** As for the type **I** states, we want to determine if the type **II** states exist by demanding regularity of the central charges associated to the states (3.11). For  $A = 1$  and  $B = 2$  in (3.11), the central charge for each state is

$$Z_1^{(II)} = e^{K/2} (p^0 z^2 z^3 - p^2 z^3 + q_1) z^1, \quad (3.16a)$$

$$Z_2^{(II)} = e^{K/2} (p^2 z^1 z^3 - q_1 z^1 - q_0). \quad (3.16b)$$

According to (3.16a)  $Z_1^{(II)}$  has a factorized form, its first factor  $z_1$  only vanishing in singular points of moduli space. If we focus on the second factor, its imaginary component allows us to solve for one of the moduli:

$$p^0 B^2 = p^2 - \frac{J^2}{J^3} B^3 p^0.$$

Substituting this into the real part of the same factor, we obtain the constraint

$$q_1 = p^0 \frac{J^2}{J^3} |z^3|^2.$$

Since volumes  $J^A$  are positive and  $|z^3|$  only vanishes at singular points in moduli space, we reach the conclusion:

1.  $Z_1^{(II)}$  has zeroes at non-singular points in moduli space *if*  $p^0 q_1 > 0$ .
2.  $Z_1^{(II)}$  has *no* zeroes at non-singular points in moduli space *if*  $p^0 q_1 < 0$ .

The analysis for  $Z_2^{(II)}$  is entirely analogous, and the conclusions similar in nature:

1.  $Z_2^{(II)}$  has zeroes at non-singular points in moduli space *if*  $p^2 q_0 < 0$ .
2.  $Z_2^{(II)}$  has *no* zeroes at non-singular points in moduli space *if*  $p^2 q_0 > 0$ .

Therefore, we have that a type **II** state will be well defined in moduli space if

$$p^0 q_1 < 0, \quad p^2 q_0 > 0. \quad (3.17)$$

Let us match these observations with the positivity of  $\Sigma^2$  close to the pole, as we did for type **I** states. Consider a state with central charge  $Z_1^{(II)}$  and charge vector  $\Gamma_1^{(II)}$ . The central charge phase satisfies

$$|Z_1^{(II)}| \sin \alpha = e^{K/2} (p^0 \text{Im}(z^1 z^2 z^3) - p^2 \text{Im}(z^1 z^3) + q_1 J^1).$$

The dominant contribution to  $\Sigma^2$  close to the charge vector location ( $\vec{x}_1$ ) is

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_1) \rightarrow \frac{16 e^K J^2 J^3 |z^1|^2}{|\vec{x} - \vec{x}_1|^2} (-p^0 q_1) + \mathcal{O}(|\vec{x} - \vec{x}_1|^{-1}) \quad (3.18)$$

The behavior is consistent with a 1/4 BPS state, as it should, and the function is positive if  $p^0 q_1 < 0$ , which matches the condition for the regularity of  $Z_1^{(II)}$ .



### 3.3 Walls of marginal stability

Having identified the potential BPS constituents for our bound states, we would like to solve the conditions (2.21) for the two possible splits: (3.10) and (3.11). These conditions describe walls of marginal stability and define the region of moduli space where the bound states exist.

**Type I split.** For this split, the D0-D6 charge vector decomposes into

$$\Gamma_{\text{D0-D6}} \rightarrow \Gamma_1^{(I)} + \Gamma_2^{(I)} , \quad (3.19)$$

with the final states carrying charges (3.10) and the central charges of each constituent are (3.12). The imaginary and real part of  $(Z_1^{(I)} \bar{Z}_2^{(I)})$  are given by

$$\begin{aligned} e^{-K} \text{Im} \left( Z_1^{(I)} \bar{Z}_2^{(I)} \right) &= -J^1 \left( ad q_0^2 - p^0 q_0 (ab + cd) \text{Re}(z^2 z^3) + bc (p^0)^2 |z^2 z^3|^2 \right) \\ &\quad - p^0 q_0 \text{Im}(z^2 z^3) \left( bc - (ab + cd) B^1 + ad |z^1|^2 \right) , \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} e^{-K} \text{Re} \left( Z_1^{(I)} \bar{Z}_2^{(I)} \right) &= - \left( bc - (ab + cd) B^1 + ad |z^1|^2 \right) \left( ad q_0^2 - p^0 q_0 (ab + cd) \text{Re}(z^2 z^3) \right) \\ &\quad + bc (p^0)^2 |z^2 z^3|^2 + p^0 q_0 \text{Im}(z^2 z^3) J^1 . \end{aligned} \quad (3.21)$$

Imposing mass conservation,  $|Z_{\text{D0-D6}}| = |Z_1^{(I)}| + |Z_2^{(I)}|$ , which is equivalent to setting (3.20) equal to zero gives

$$\begin{aligned} &J^1 \left( ad q_0^2 - p^0 q_0 (ab + cd) \text{Re}(z^2 z^3) + bc (p^0)^2 |z^2 z^3|^2 \right) \\ &= -p^0 q_0 \text{Im}(z^2 z^3) \left( bc - (ab + cd) B^1 + ad |z^1|^2 \right) . \end{aligned} \quad (3.22)$$

In addition, according to (2.21) the phases will be aligned along the wall if  $\text{Re} Z_1^{(I)} \bar{Z}_2^{(I)} > 0$ , which reduces to

$$p^0 q_0 \text{Im}(z^2 z^3) J^1 > 0 , \quad (3.23)$$

where we used (3.21) and (3.22). Since  $J^1$  is always positive and non-zero, (3.23) becomes

$$p^0 q_0 \text{Im}(z^2 z^3) > 0 . \quad (3.24)$$

Equation (3.22) describes circles or straight lines in the  $z^1$  complex plane for constant  $(z^2 z^3)$ . These circles are exactly those found in [5, 7, 42, 43, 44], where the analysis was done for 1/4 BPS states in  $\mathcal{N} = 4$  theory decaying into two 1/2 BPS states. Here  $z^1$  can be interpreted as the axion-dilaton moduli. The charges vectors (3.10) can be written as electric  $Q$  and magnetic  $P$  vectors of the  $O(6, n)$  duality group of  $\mathcal{N} = 4$ . For example, if  $A = 1$  in (3.10a) the D-brane charges correspond in the Heterotic frame to [45]

$$\begin{aligned} Q^{(I)} &= (cdq_0, -bcp^0, \vec{0}) , \\ P^{(I)} &= (adq_0, -abp^0, \vec{0}) , \end{aligned} \quad (3.25)$$

and a similar expression for (3.10b). This is what we would expect for 1/2 BPS states in  $\mathcal{N} = 4$ , since the electric and magnetic vectors in (3.25) are parallel.

**Type II split.** We proceed to determine the marginal stability condition for D0-D6 when final states carry the charges in (3.11). For simplicity, we re-write the central charge of each constituent (3.16) as

$$Z_1^{(II)} = e^{K/2} (p^0 z^1 z^2 z^3 - Y) , \quad (3.26a)$$

$$Z_2^{(II)} = e^{K/2} (Y - q_0) . \quad (3.26b)$$

with

$$Y \equiv p^2 z^1 z^3 - q_1 z^1 . \quad (3.27)$$

The imaginary and real part of  $(Z_1^{(II)} \bar{Z}_2^{(II)})$  are

$$e^{-K} \text{Im} \left( Z_1^{(II)} \bar{Z}_2^{(II)} \right) = \text{Im} Y (q_0 - p^0 \text{Re}(z^1 z^2 z^3)) + p^0 \text{Im}(z^1 z^2 z^3) (\text{Re} Y - q_0) , \quad (3.28)$$

and

$$e^{-K} \text{Re} \left( Z_1^{(II)} \bar{Z}_2^{(II)} \right) = - (q_0 - p^0 \text{Re}(z^1 z^2 z^3)) (\text{Re} Y - q_0) + p^0 \text{Im}(z^1 z^2 z^3) \text{Im} Y - |Y - q_0|^2 . \quad (3.29)$$

The first condition of marginal stability in (2.21) simplifies to

$$\text{Im} Y (q_0 - p^0 \text{Re}(z^1 z^2 z^3)) + p^0 \text{Im}(z^1 z^2 z^3) (\text{Re} Y - q_0) = 0 , \quad (3.30)$$

which imposes mass conservation. The phase of each state will be align along the wall (3.30) when

$$|Y - q_0|^2 \left( -1 + \frac{p^0 \text{Im}(z^1 z^2 z^3)}{\text{Im} Y} \right) > 0 . \quad (3.31)$$

The two conditions, (3.30) and (3.31), define the wall of marginal stability for type **II** bound states. In the following section, we will investigate if the conditions found in this section for type **I** and **II** splits are sufficient or just necessary for the state to have a well-behaved supergravity description.

## 4 Bound states as two-centered solutions

We will now examine whether the actual bound state, when realized as a two-centered supergravity configuration, is a regular configuration. Previously, we established a set of possible charge splits of the total D0-D6 charge vector consistent with supersymmetry, and we described the regions of moduli space where the individual and bound BPS states exist by imposing local algebraic conditions. In the following we will study global conditions on the geometry to assure the existence of the bound state.

#### 4.1 D0 and D6 constituents

To illustrate the procedure we start with the simplest bound state, *i.e.*  $\Gamma_{\text{D0-D6}} = \Gamma_{\text{D0}} + \Gamma_{\text{D6}}$ . This corresponds to  $c = d = 0$  and  $ab = -1$  in (3.10). The constituent central charges are

$$Z_{\text{D6}} = e^{K/2} p^0 z^1 z^2 z^3, \quad Z_{\text{D0}} = -e^{K/2} q_0. \quad (4.1)$$

Both quantities are regular in non-singular points of moduli space. The metric and one form  $\omega$  are as discussed in section 2.1.1, and the helicity of the state is  $\langle \Gamma_{\text{D6}}, \Gamma_{\text{D0}} \rangle = -p^0 q_0$ . We choose the D6-branes to be located at the origin  $\vec{x}_1 = \vec{0}$  with charge  $p^0$ , and the D0 branes at  $\vec{x}_2 = (0, 0, R)$  with charge  $q_0$ . The set of harmonic functions are

$$H^0 = h^0 + \frac{p^0}{r}, \quad H_0 = h_0 + \frac{q_0}{\Theta}, \quad H_A = h_A, \quad H^A = h^A, \quad (4.2)$$

with  $r$  and  $\Theta$  defined by (2.13). The integrability conditions (2.14) reduce to

$$p^0 h_0 = q_0 h^0, \quad \frac{p^0 q_0}{R} = -h^0 q_0. \quad (4.3)$$

Using the moduli identities listed in appendix C and the integrability conditions (4.3), the function (2.6) reads

$$\begin{aligned} \Sigma^2(H) = & -\frac{1}{r^2} \left( \frac{p^0 q_0}{R} \right)^2 \left[ 1 + \frac{r-R}{\Theta} \right]^2 + \frac{4}{r\Theta} (p^0 q_0) (h^1 h_1 + 4e^K B^1 J^2 J^3) \\ & + \frac{4}{|Z_{\text{D0D6}}|} \left[ \frac{1}{\Theta} \text{Re}(Z_{\text{D0-D6}} \bar{Z}_{\text{D6}}) + \frac{1}{r} \text{Re}(Z_{\text{D0-D6}} \bar{Z}_{\text{D0}}) \right] + 1. \end{aligned} \quad (4.4)$$

The existence of the bound state requires that (4.4) is positive definite throughout spacetime. In particular, close to each center we have

$$\begin{aligned} \Sigma^2(\vec{x} \rightarrow \vec{x}_1) &= -\frac{4p^0 h_1}{r} \left( -\frac{q_0}{R} h^1 + 4e^K |z^1|^2 J^2 J^3 \right) + \dots, \\ \Sigma^2(\vec{x} \rightarrow \vec{x}_2) &= -\frac{4q_0 h^1}{\Theta} \left( -\frac{p^0}{R} h_1 + 4e^K J^2 J^3 \right) + \dots, \end{aligned} \quad (4.5)$$

where the dots denote subleading terms. Notice the divergence at each center is consistent with having a 1/2 BPS charge vector constituent, but the actual coefficient does depend on the moduli and the *total* central charge phase  $\alpha$ . Contrary to what occurs for single centered 1/2 BPS supergravity configurations in (3.14) and (3.15), the above expressions are not positive definite for *any* value of the moduli and  $\alpha$ . The analysis of marginal stability in section 3.3, showed that the phases of the central charges are aligned if  $p^0 q_0 \text{Im}(z^2 z^3) > 0$ . Combining this with  $J^A > 0$  and (C.2) tells us that

$$-h^1 q_0 > 0 \quad -p^0 h_1 > 0. \quad (4.6)$$

Therefore, the near pole behavior (4.5) is positive in the same region of moduli space described by the conditions of existence of the bound state in the previous section.

Further, one can prove the absence of CTCs in the full geometry by proving that (2.17) is satisfied everywhere. For the D0-D6 bound state we have

$$\begin{aligned} \Sigma^2 r^2 \sin^2 \theta - (\omega_\phi)^2 = \sin^2 \theta & \left[ \frac{4r}{\Theta} p^0 q_0 e^K (\text{Im}(z^2 z^3) J^1 + B^1 J^2 J^3) \right. \\ & + \frac{4r^2}{|Z_{\text{D0D6}}|} \left( \frac{1}{\Theta} \text{Re}(Z_{\text{D0-D6}} \bar{Z}_{\text{D6}}) + \frac{1}{r} \text{Re}(Z_{\text{D0-D6}} \bar{Z}_{\text{D0}}) \right) + r^2 \Big] \\ & + \frac{4}{\Theta} \left( \frac{p^0 q_0}{R} \right)^2 (1 - \cos \theta)(r + R - \Theta) . \end{aligned} \quad (4.7)$$

Each term in (4.7) is positive definite in the region of moduli space defined by  $\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(Z_1 \bar{Z}_2) > 0$  and  $\text{Re}(Z_1 \bar{Z}_2) > 0$ .<sup>8</sup> Thus, the conditions of marginal stability are sufficient for a regular two-centered solution to exist with D0 and D6 charge split.

In the remaining of this section we will study the regularity of the supergravity configurations describing the more general type **I** and type **II** split states identified before. The tools and methodology are the same as for the D0-D6. We will argue that for only very specific cases the conditions of marginal stability (2.21) are sufficient to guarantee regularity of the two-centered solution.

## 4.2 Type I bound states

Consider a two-centered configuration with centers  $\vec{x}_1 = \vec{0}$  and  $\vec{x}_2 = (0, 0, R)$  carrying charges  $\Gamma_1^{(I)}$  and  $\Gamma_2^{(I)}$ , respectively. For simplicity, we will set  $A = 1$  in (3.10). The set of harmonic functions is given by

$$\begin{aligned} H^0 &= h^0 - \frac{ab p^0}{r} + \frac{cd p^0}{\Theta} , & H^1 &= h^1 - \frac{bc p^0}{r} + \frac{bc p^0}{\Theta} , \\ H_0 &= h_0 + \frac{cd q_0}{r} - \frac{ab q_0}{\Theta} , & H_1 &= h_1 + \frac{ad q_0}{r} - \frac{ad q_0}{\Theta} , \end{aligned}$$

with  $cd - ab = 1$  and  $\Theta^2 = r^2 + R^2 - 2rR \cos \theta$ . The remaining harmonic functions are constant, *i.e.*  $H^{2,3} = h^{2,3}$  and  $H_{2,3} = h_{2,3}$ . The factor (2.6) is

$$\begin{aligned} \Sigma^2(H) &= - \left( -H_0 H^0 + H_1 H^1 + h_2 h^2 + h_3 h^3 \right)^2 + 4H^1 H_1 (h^2 h_2 + h^3 h_3) \\ &\quad - 4H^0 H_1 h_2 h_3 - 4H_0 H^1 h^2 h^3 + 4h^2 h_2 h^3 h_3 . \end{aligned} \quad (4.8)$$

The integrability conditions (2.14) read

$$q_0 h^0 = p^0 h_0 , \quad \frac{p^0 q_0}{R} = -q_0 h^0 + \frac{1}{ab + cd} (ad q_0 h^1 + bc p^0 h_1) , \quad (4.9)$$

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<sup>8</sup>The first term proportional to  $p^0 q_0 (\text{Im}(z^2 z^3) J^1 + B^1 J^2 J^3)$  can be shown to be positive by assuming it is negative and then showing such an assumption is not consistent with  $\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(Z_1 \bar{Z}_2) > 0$  and  $\text{Re}(Z_1 \bar{Z}_2) > 0$ .

whereas the helicity of the state is given by

$$\langle \Gamma_1^{(I)}, \Gamma_2^{(I)} \rangle = (ab + cd)p^0 q_0 . \quad (4.10)$$

Now we proceed to study the positivity of (4.8). As  $\vec{x} \rightarrow \infty$  the metric is asymptotically flat, therefore  $\Sigma^2 \rightarrow 1$ . Close to each pole  $\{\vec{x}_1, \vec{x}_2\}$  it should remain positive in order to avoid fake horizons. In the limit  $\vec{x} \rightarrow \vec{x}_1$ , the leading terms in (4.8) are

$$\begin{aligned} \Sigma^2(\vec{x} \rightarrow \vec{x}_1) = & -\frac{1}{r^2} \left( -q_0 h^0 - \frac{p^0 q_0}{R} + ad q_0 h^1 - bc p^0 h_1 \right)^2 - \frac{4abcd p^0 q_0}{r^2} (h^2 h_2 + h^3 h_3) \\ & + \frac{4a^2 bd p^0 q_0}{r^2} h_2 h_3 + \frac{4b^2 cd p^0 q_0}{r^2} h^2 h^3 + \mathcal{O}\left(\frac{1}{r}\right) . \end{aligned} \quad (4.11)$$

Using (4.9) and after some algebra, we can rewrite (4.11) as

$$\begin{aligned} \Sigma^2(\vec{x} \rightarrow \vec{x}_1) = & -\frac{4bd p^0 q_0}{r^2} (a^2 q_0 h^1 + c^2 p^0 h_1) \frac{1}{R} + \frac{4abcd}{r^2} \left( \frac{p^0 q_0}{R} + h^0 q_0 \right)^2 \\ & + \frac{4bd p^0 q_0}{r^2} [a^2 (h_2 h_3 - h_0 h^1) + c^2 (h^2 h^3 - h^0 h_1) - ac (h^2 h_2 + h^3 h_3 - h_1 h^1)] \\ & + \mathcal{O}\left(\frac{1}{r}\right) . \end{aligned} \quad (4.12)$$

Notice the dependence on the moduli and the total central charge phase  $\alpha$  is very different from the one we found for the single centered solution with the same center vector charge in (3.14). This is because of the singular nature of these solutions to the attractor equations. Since  $\Delta = 0$ , the dominant (non-vanishing) contribution to  $\Sigma^2$  is not fixed by the attractor mechanism, and as such, it depends on global aspects of the solution. From this perspective, the positivity of  $\Sigma^2$  at each center is already a non-trivial condition for the bound state to exist.

Analogously, the behavior of (4.8) close to the second center is

$$\begin{aligned} \Sigma^2(\vec{x} \rightarrow \vec{x}_2) = & -\frac{4ac p^0 q_0}{\Theta^2} (b^2 q_0 h^1 + d^2 p^0 h_1) \frac{1}{R} + \frac{4abcd}{\Theta^2} \left( \frac{p^0 q_0}{r_{12}} + h^0 q_0 \right)^2 \\ & + \frac{4ac p^0 q_0}{\Theta^2} [d^2 (h_2 h_3 - h_0 h^1) + b^2 (h^2 h^3 - h^0 h_1) - bd (h^2 h_2 + h^3 h_3 - h^1 h_1)] \\ & + \mathcal{O}\left(\frac{1}{\Theta}\right) . \end{aligned} \quad (4.13)$$

From the condition of marginal stability, we found that the bound state will exist when (3.24) holds. Combining this condition with the fact that  $J^A > 0$  in (C.2), we have

$$-h^1 q_0 > 0 \quad -p^0 h_1 > 0 . \quad (4.14)$$

Therefore the first and second term in (4.12) and (4.13) will be positive if  $bd p^0 q_0 > 0$  and  $ac p^0 q_0 > 0$ . This is consistent with the condition (3.13) derived by imposing regularity of the central charge vectors.

Using (4.9), (C.3) and (C.4), we can write the last term in (4.12) and (4.13) as

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_1) = \dots + \frac{4bd p^0 q_0}{r^2} e^K J^2 J^3 |az_1 - c|^2 - \frac{4abcd}{r^2} (h^0 q_0)^2 + \mathcal{O}\left(\frac{1}{r}\right) , \quad (4.15a)$$

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_2) = \dots + \frac{4ac p^0 q_0}{\Theta^2} e^K J^2 J^3 |dz_1 - b|^2 - \frac{4abcd}{\Theta^2} (h^0 q_0)^2 + \mathcal{O}\left(\frac{1}{\Theta}\right) . \quad (4.15b)$$

The first term for both poles is also positive if  $bdp^0 q_0 > 0$  and  $acp^0 q_0 > 0$ , but the second term is negative for this assignment of charges. This tell us that in order to have  $\Sigma^2 > 0$  for  $(a, b, c, d)$  non-zero we need to impose further constraints on the moduli, which will raise the co-dimension of the walls of marginal stability. This conclusion can be avoided if we have one (or two) vanishing coefficients among  $(a, b, c, d)$  while still satisfying  $cd - ab = 1$ . In these cases, the bound state may still exist. We will explore in more detail this scenario in the remaining of this section.

Before proceeding, let us emphasize that at this point we have already established the existence of further requirements beyond supersymmetry, regularity of the central charge and existence of a wall of marginal stability for the supergravity supersymmetric bound state to exist. From a purely supergravity perspective, this also provides an example for families of configurations that solve the integrability conditions but are *not* free of CTCs.

#### 4.2.1 Surviving Type I states

For non-zero values of  $(a, b, c, d)$ , we found in (4.15) that the conditions of marginal stability are not sufficient to assure a positive  $\Sigma^2$  close to each pole. But if one of the integers is zero, the negative contribution in (4.15) vanishes. In the following, we will study the regularity of the supergravity solutions for such configurations. Consider

$$\Gamma_1 = (p^0, [-p]; 0, 0) , \quad \Gamma_2 = (0, [p]; 0, q_0) , \quad (4.16)$$

where the first vector corresponds to a D6 brane ( $p^0$ ) and an anti-D4 wrapping a 4-cycle of  $T^6$  with charge  $-p$ , and the vector  $\Gamma_2$  corresponds to a D0 brane ( $q_0$ ) and a D4 wrapping the same cycle. The other possible combination is

$$\Gamma_1 = (p^0, 0; [q], 0) , \quad \Gamma_2 = (0, 0; [-q], q_0) , \quad (4.17)$$

where the first vector corresponds to a D6 brane and a D2 wrapping a 2-cycle of  $T^6$  with charge  $q$ , and the vector  $\Gamma_2$  corresponds to a D0 brane and an anti-D2 wrapping the same cycle. Using the notation in (3.10), states (4.16) correspond to  $d = 0$  and  $p \equiv bcp^0$ , and states (4.17) correspond to  $c = 0$  and  $q \equiv adq^0$ .

The analysis of regularity for both configurations (4.16) and (4.17) is completely analogous. For brevity, we will carry the analysis only for (4.16). First consider the conditions of marginal stability. The central charges of each state is given by

$$Z_1 = e^{K/2} (p^0 z^1 + p) z^2 z^3 , \quad Z_2 = -e^{K/2} (p z^2 z^3 + q_0) . \quad (4.18)$$

Demanding regularity of the central charges requires

$$p q_0 < 0 . \quad (4.19)$$

The bound state is stable if

$$\langle \Gamma_1, \Gamma_2 \rangle \text{Im} (Z_1 \bar{Z}_2) > 0 , \quad \text{Re} (Z_1 \bar{Z}_2) > 0 . \quad (4.20)$$

Inserting (4.18) in the above conditions<sup>9</sup> gives

$$\text{Im} z^1 \text{Re}(z^2 z^3) + \text{Im}(z^2 z^3) \text{Re} z^1 + \frac{p}{p^0} \text{Im}(z^2 z^3) + \frac{p}{q_0} |x|^2 \text{Im} z^1 > 0 , \quad (4.21)$$

and

$$p^0 q_0 \text{Im}(z^2 z^3) > 0 . \quad (4.22)$$

We proceed now to investigate the regularity conditions of the supergravity solution. One important requirement is the absence of closed timelike curves

$$\Sigma^2 r^2 \sin^2 \theta - (\omega_\phi)^2 > 0 . \quad (4.23)$$

If (4.23) is satisfied this will also imply that  $\Sigma^2$  is positive through out the geometry. For the solution in hand, the metric factor (4.8) is

$$\begin{aligned} \Sigma^2(H) = 1 + \frac{4}{|Z_{D0D6}|} & \left( \frac{1}{r} \text{Re}(Z_1 \bar{Z}_{D0-D6}) + \frac{1}{\Theta} \text{Re}(Z_2 \bar{Z}_{D0-D6}) \right) \\ & + \frac{4}{r\Theta} p^0 q_0 e^K \left( \text{Im}(z^2 z^3) J^1 + B^1 J^2 J^3 + \frac{p}{p^0} J^2 J^3 \right) + \frac{4}{r\Theta^2} (p q_0) p^0 h_1 \\ & + \frac{4}{r\Theta} \left( \frac{p^0 q_0}{R} \right)^2 - \frac{1}{r^2} \left( \frac{p^0 q_0}{R} \right)^2 \left( 1 + \frac{r-R}{\Theta} \right)^2 \end{aligned} \quad (4.24)$$

where  $Z_{1,2}$  are defined by (4.18). The one-form rotation is given by (2.16) and for the charges (4.16) it reads

$$\omega = -\frac{p^0 q_0}{R} \left[ 1 - \frac{r+R}{\Theta} \right] (1 - \cos \theta) d\phi . \quad (4.25)$$

Inserting (4.24) and (4.25) in (4.23) we get

$$\begin{aligned} \Sigma^2 r^2 \sin^2 \theta - (\omega_\phi)^2 = & r^2 \sin^2 \theta \left[ 1 + \frac{4}{|Z_{D0D6}|} \left( \frac{1}{r} \text{Re}(Z_1 \bar{Z}_{D0-D6}) + \frac{1}{\Theta} \text{Re}(Z_2 \bar{Z}_{D0-D6}) \right) \right. \\ & + \frac{4}{r\Theta} p^0 q_0 e^K \left( \text{Im}(z^2 z^3) J^1 + B^1 J^2 J^3 + \frac{p}{p^0} J^2 J^3 \right) + \frac{4}{r\Theta^2} (p q_0) p^0 h_1 \\ & \left. + \frac{4}{\Theta} \left( \frac{p^0 q_0}{R} \right)^2 (1 - \cos \theta)(r + R - \Theta) \right] \end{aligned}$$

All terms are positive definite. Thus, these configurations are free of CTCs. As a consequence of this derivation,  $\Sigma^2$  is positive everywhere, and we conclude the supergravity realization of the supersymmetric bound state exists.

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<sup>9</sup>Or equivalently setting  $d = 0$  in (3.22) and (3.24).

### 4.3 Type II split

The discussion is analogous to type I. The bound state should be a two-centered solution with centers  $\vec{x}_1$  and  $\vec{x}_2$  carrying charges  $\Gamma_1^{(II)}$  and  $\Gamma_2^{(II)}$  given by (3.11). The set of harmonic functions are given by

$$\begin{aligned} H^0 &= h^0 + \frac{p^0}{|\vec{x} - \vec{x}_1|} , & H^2 &= h^2 + \frac{p^2}{|\vec{x} - \vec{x}_1|} - \frac{p^2}{|\vec{x} - \vec{x}_2|} , \\ H_0 &= h_0 + \frac{q_0}{|\vec{x} - \vec{x}_2|} , & H_1 &= h_1 + \frac{q_1}{|\vec{x} - \vec{x}_1|} - \frac{q_1}{|\vec{x} - \vec{x}_2|} . \end{aligned}$$

The remaining harmonic functions are constant, *i.e.*  $H^{1,3} = h^{1,3}$  and  $H_{2,3} = h_{2,3}$ . From (2.6), the metric factor for this bound state is

$$\begin{aligned} \Sigma^2(H) &= - \left( -H_0 H^0 + H_1 h^1 + h_2 H^2 + h_3 h^3 \right)^2 + 4h^1 H_1 (H^2 h_2 + h^3 h_3) \\ &\quad - 4H^0 H_1 h_2 h_3 - 4H_0 h^1 H^2 h^3 + 4H^2 h_2 h^3 h_3 . \end{aligned} \quad (4.26)$$

For the charge vectors (3.11), the helicity of the state is

$$\langle \Gamma_1^{(II)}, \Gamma_2^{(II)} \rangle = -p^0 q_0 , \quad (4.27)$$

and the integrability conditions (2.14) reduce to

$$q_0 h^0 = p^0 h_0 , \quad \frac{p^0 q_0}{R} = -q_0 h^0 - q_1 h^1 + p^2 h_2 . \quad (4.28)$$

As before, our first check is to study the positivity of  $\Sigma^2$  close to each pole. In the limit  $\vec{x} \rightarrow \vec{x}_1$  and  $\vec{x} \rightarrow \vec{x}_2$ , the leading terms in (4.26) are

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_1) = -\frac{4q_1 p^0}{|\vec{x} - \vec{x}_1|^2} \left( 4e^K |z^1|^2 J^2 J^3 - \frac{q_0 h^1}{R} \right) + \mathcal{O}(|\vec{x} - \vec{x}_1|^{-1}) , \quad (4.29a)$$

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_2) = \frac{4q_0 p^2}{|\vec{x} - \vec{x}_2|^2} \left( 4e^K J^2 J^3 - \frac{p^0 h_2}{R} \right) + \mathcal{O}(|\vec{x} - \vec{x}_2|^{-1}) . \quad (4.29b)$$

where we used (4.28) and (C.3). From section 3.2, the central charges  $Z_1^{(II)}$  and  $Z_2^{(II)}$  are regular if

$$p^0 q_1 < 0 , \quad p^2 q_0 > 0 , \quad (4.30)$$

hence the first term in each parenthesis in (4.29) is positive. The second term in (4.29a) gives

$$q_0 p^0 q_1 h^1 = \frac{2e^{K/2}}{|Z_{D0-D6}|} \left( -p^0 q_1 (q_0)^2 J^1 - p^0 q_1 (q_0 p^0) |z^1|^2 \text{Im}(z^2 z^3) \right) . \quad (4.31)$$

Is this quantity positive? From the analysis of the central charges and the integrability conditions, the stable region for the state is defined by  $R > 0$  in (4.28) and delimited by the walls of marginal stability (3.30) and (3.31). These conditions are not sufficient for having (4.31) positive definite. Analogously, by studying (4.29b) we reach the same result. Therefore we conclude that  $\Sigma^2$  can be negative close to the poles unless we impose further constraints on the moduli, increasing the co-dimension of the walls of marginal stability.



## 5 Bound states including 1/8 BPS states

In previous sections, we studied the supersymmetric D0-D6 bound states as supergravity two-centered configurations involving 1/4 and 1/2 BPS charge vectors. In principle, it is also possible to include as constituents 1/8 BPS states with vanishing quartic invariant.<sup>10</sup> Here we will argue that regularity of the solution will generically impose further constraints on the moduli. Thus, if it exists, it will do so in a region of moduli space of co-dimension higher than one. Our strategy consists on studying the behavior of  $\Sigma$  for a generic two-centered solution, where one of the centers is a 1/8 BPS state with vanishing quartic invariant. Close to this center the positivity of  $\Sigma^2$  is not guaranteed by the integrability conditions, hence generically there will be additional restrictions on the moduli.

Consider a two-centered supergravity configuration such that  $\Gamma_{\text{D0-D6}} = \Gamma_1 + \Gamma_2$ . The pole at  $\vec{x}_1$  carries a charge vector  $\Gamma_1 = (p_1^\Lambda, q_1^\Lambda)$  corresponding to a 1/8 BPS state with vanishing quartic invariant. Thus,  $\Delta_1 = 0$  and at least one  $\partial\Delta_1/\partial p^\Lambda$  and/or  $\partial\Delta_1/\partial q_A$  are non-vanishing. The second pole  $\vec{x}_2$  carries charge  $\Gamma_2 = (p_2^\Lambda, q_2^\Lambda)$ .<sup>11</sup> The behavior of the function  $\Sigma^2$  close to the center  $\vec{x}_1$  is

$$\begin{aligned} \Sigma^2(\vec{x} \rightarrow \vec{x}_1) \rightarrow \frac{1}{|\vec{x} - \vec{x}_1|^3} & \left( \frac{\partial\Delta_1}{\partial p_1^0} \left( h^0 + \frac{p_2^0}{R} \right) + \frac{\partial\Delta_1}{\partial p_1^A} \left( h^A + \frac{p_2^A}{R} \right) \right. \\ & \left. + \frac{\partial\Delta_1}{\partial q_0^1} \left( h_0 + \frac{q_0^2}{R} \right) + \frac{\partial\Delta_1}{\partial q_A^1} \left( h_A + \frac{q_A^2}{R} \right) \right) + \mathcal{O}(|\vec{x} - \vec{x}_1|^{-2}) . \end{aligned} \quad (5.1)$$

Given its linear dependence on  $(h^\Lambda; h_\Lambda)$ , we can use the integrability condition (2.14) fixing the distance scale  $R$  between the two centers and the definitions given in (A.3) to rewrite this expression as

$$\Sigma^2(\vec{x} \rightarrow \vec{x}_1) \rightarrow \frac{2 \operatorname{Im} (Z_\star \bar{Z}_{\text{D0-D6}})}{|Z_{\text{D0-D6}}| \langle \Gamma_1, \Gamma_2 \rangle} \frac{1}{|\vec{x} - \vec{x}_1|^3} + \mathcal{O}(|\vec{x} - \vec{x}_1|^{-2}) , \quad (5.2)$$

where  $Z_\star$  is the central charge associated with the effective charge vector

$$\Gamma_\star = \langle \Gamma_1, \Gamma_2 \rangle \Gamma_{\text{eff}} - \langle \Gamma_{\text{eff}}, \Gamma_2 \rangle \Gamma_1 , \quad (5.3)$$

with

$$\Gamma_{\text{eff}} \equiv \left( \frac{\partial\Delta_1}{\partial q_0^1}, -\frac{\partial\Delta_1}{\partial q_A^1}, \frac{\partial\Delta_1}{\partial p_1^A}, -\frac{\partial\Delta_1}{\partial p_1^0} \right) .$$

Thus, positivity of  $\Sigma^2$  in this limit requires

$$\langle \Gamma_1, \Gamma_2 \rangle \operatorname{Im} (Z_\star \bar{Z}_{\text{D0-D6}}) > 0 . \quad (5.4)$$

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<sup>10</sup>The possibility of allowing 1/8 BPS states with positive quartic invariant is entropically disfavored, and we will not consider it here.

<sup>11</sup>Since  $\Gamma_2$  is supersymmetric and has vanishing quartic invariant,  $\Delta_2 = 0$ , conservation of charge puts some non-trivial constraints on its components. We will not need these details here, though this is a problem that can be solved.

Generically, this imposes a condition on the relative phases of both central charges, which is moduli dependent.

Let us assume the existence of a supersymmetric bound state in a region of moduli space bounded by a wall of marginal stability of co-dimension one. This requires the following conditions to hold

$$\langle \Gamma_1, \Gamma_{D0-D6} \rangle \text{Im} (Z_1 \bar{Z}_{D0-D6}) > 0 \quad \text{and} \quad \text{Re} (Z_1 \bar{Z}_2) > 0. \quad (5.5)$$

The question is whether (5.5) guarantees the positivity of  $\Sigma^2$  at the center  $\vec{x}_1$  without introducing any further constraint on the moduli, *i.e.* if (5.4) is consistent with (5.5). A subset of effective central charges  $Z_\star$  that would trivially satisfy this property would be

$$Z_\star = (\beta + i \langle \Gamma_1, \Gamma_2 \rangle \gamma) Z_1 + (\alpha + i \langle \Gamma_1, \Gamma_2 \rangle \delta) Z_2,$$

$\forall \alpha < 0$  and  $\forall \beta, \gamma, \delta > 0$ . This imposes a condition on the effective charge vector  $\Gamma_\star$ ,<sup>12</sup>

$$\Gamma_\star = (\beta + i \langle \Gamma_1, \Gamma_2 \rangle \gamma) \Gamma_1 + (\alpha + i \langle \Gamma_1, \Gamma_2 \rangle \delta) \Gamma_2, \quad (5.6)$$

which is a non-linear equation to be satisfied for the charge components of the original 1/8 BPS state. Since this is an equality between charge vectors, we can check its consistency with charge conservation by computing its inner product with  $\Gamma_1$  and  $\Gamma_2$ . Using the fact that  $\langle \Gamma_{\text{eff}}, \Gamma_1 \rangle = -4\Delta_1 = 0$ , we learn from (5.3) that

$$\langle \Gamma_1, \Gamma_\star \rangle = 0,$$

and so the inner product of (5.6) with  $\Gamma_1$  gives rise to

$$0 = (\alpha + i \langle \Gamma_1, \Gamma_2 \rangle \delta) \langle \Gamma_1, \Gamma_2 \rangle.$$

Thus, for mutually non-local charge vectors,  $\alpha = \delta = 0$ . Similarly, computing the inner product with  $\Gamma_2$  and using the antisymmetry properties of it, we get

$$0 = (\beta + i \langle \Gamma_1, \Gamma_2 \rangle \gamma) \langle \Gamma_1, \Gamma_2 \rangle.$$

Once again, for mutually non-local charge vectors, we must conclude  $\beta = \gamma = 0$ . All in all, we learn that there is no  $Z_\star$  trivially satisfying (5.4), being consistent with charge conservation and having a 1/8 BPS constituent with vanishing quartic invariant. Any other choice of  $Z_\star$  would give rise to a further constraint on the moduli.

We conclude that any pair of charge vectors  $\{\Gamma_1, \Gamma_2\}$  with  $\Gamma_1$  being 1/8 BPS with  $\Delta_1 = 0$ , consistent with supersymmetry and charge conservation will have some extra moduli dependent condition ensuring the positivity of  $\Sigma^2$  close to the 1/8 BPS center  $\vec{x}_1$  and necessarily increasing the co-dimension of its wall of marginal stability.

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<sup>12</sup> $\Gamma_\star$  does not have to correspond to any physical charge in principle. It is just a convenient mathematical way of encoding the behavior of  $\Sigma^2$  near the pole  $\vec{x}_1$ .

## 6 Discussion

We studied the gravitational realization of supersymmetric D0-D6 bound states in the STU model. In the large volume limit, we determined all supersymmetric regular two-centered configurations consistent with the composites of the system existing in regions of moduli space bounded by a wall of marginal stability of co-dimension one. The possible constituents states of the system are

$$\begin{aligned}\Gamma_1 &= (p^0, [-p]; 0, 0) , & \Gamma_2 &= (0, [p]; 0, q_0) , \\ \Gamma_1 &= (p^0, 0; [q], 0) , & \Gamma_2 &= (0, 0; [-q], q_0) .\end{aligned}\tag{6.1}$$

The domain in moduli space where the bound state exists is described by (3.22) and (3.24). The shape of these walls is analogous to those first found in [5, 7]. At this level,  $p$  and  $q$  are only constrained by our discussion in section 3.2. After imposing charge quantization on the vectors (6.1), *i.e.* discrete U-duality group, the final states (6.1) will be further reduced.

We have explicitly seen how global requirements of regularity imposed additional constraints on the existence of the state, besides the more kinematical (or algebraic) characterization of the charge vectors and their central charges. In other words, the local conditions from supersymmetry and regularity of the central charge are necessary but *not* sufficient to provide a well-defined supergravity configuration.

An intuitive explanation for this fact is that all allowed constituents for the system have vanishing quartic invariant. As such, they are singular solutions to the attractor equations. Whenever each of these builds a bound state, the dominant contribution to the behavior of the metric close to the center where such charge sits is no longer determined purely in terms of the charges. In addition it also depends on the moduli and the phase of the overall central charge of the bound state, which means that positivity of  $\Sigma$  in that location is already a non-trivial requirement. Indeed, we have seen that only for certain constituents such behavior is guaranteed to be positive whenever we are in the appropriate side of the wall of marginal stability, *i.e.* whenever the bound state was algebraically supposed to exist. Interestingly, whenever this requirement is fulfilled, we can also prove that the solution is free of CTCs. This observation will also be relevant for any multi-center configuration built of constituents having vanishing quartic invariants.

It would be interesting to extend our results to the full  $\mathcal{N} = 8$  theory. The additional moduli of  $E_7$  will likely impose additional constraints on the phases of the central charge [46]. It is also clearly meaningful to apply our techniques to more general situations involving polar states with  $\Delta > 0$  and attempting to relate them to the attractor flow conjecture and entropy enigma presented in [18].

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## A Multi-centered solutions in the STU-model

When  $g_s|\Gamma| \gg 1$ , we expect the supergravity approximation to provide a reliable description of any state in the theory. As reviewed in [18], the exact description will depend on the existence of the state in moduli space. More precisely, if the central charge  $Z(\Gamma)$  corresponding to a given charge vector  $\Gamma$  vanishes at a regular point in moduli space, the single centered supergravity solution will *not* exist. This is indeed the case for  $\Gamma_{D0-D6}$ . In such situations, these states can be realized in terms of multi-centered supergravity configurations, which are stationary but non-static.

In the following, we will present a very brief review of the relevant multi-centered black hole solutions constructed in [11, 12, 13]. A more recent discussion can be found in [18, 47]. The four-dimensional metric, gauge fields and moduli are given by

$$\begin{aligned}
ds^2 &= -\frac{1}{\Sigma}(dt + \omega)^2 + \Sigma ds_{\mathbb{R}^3}^2, \\
\mathcal{A}^0 &= \frac{\partial \log \Sigma}{\partial H_0}(dt + \omega) + \omega_0, \\
\mathcal{A}^A &= \frac{\partial \log \Sigma}{\partial H_A}(dt + \omega) + \mathcal{A}_d^A, \\
z^A &= \frac{H^A - i \frac{\partial \Sigma}{\partial H_A}}{H^0 + i \frac{\partial \Sigma}{\partial H_0}},
\end{aligned} \tag{A.1}$$

where  $H = (H^\Lambda; H_\Lambda)$  is a set of harmonic functions in  $\mathbb{R}^3$  which encodes the location of charges at each center. Explicitly we have

$$H^\Lambda = \sum_{i=1}^N \frac{p_i^\Lambda}{|\vec{x} - \vec{x}_i|} + h^\Lambda, \tag{A.2a}$$

$$H_\Lambda = \sum_{i=1}^N \frac{q_\Lambda^i}{|\vec{x} - \vec{x}_i|} + h_\Lambda, \tag{A.2b}$$

with  $N$  the total number of centers. A priori, it is allowed to have an arbitrary number of centers  $\vec{x}_i$  carrying charges  $\Gamma_i = (p_i^\Lambda; q_i^\Lambda)$ . The vector  $h = (h^\Lambda; h_\Lambda)$  stands for constants characterizing the asymptotic value of all the harmonic functions. More explicitly, it is given in terms of the asymptotic moduli and the phase  $\alpha$  of the total central charge by

$$\begin{aligned} h^0 &= -2e^{K/2} \sin \alpha, \\ h^A &= 2e^{K/2} (\cos \alpha \operatorname{Im}(z^A) - \sin \alpha \operatorname{Re}(z^A)), \\ h_A &= 2e^{K/2} \left( \cos \alpha \operatorname{Im} \left( \frac{1}{2} s_{ABC} z^B z^C \right) - \sin \alpha \operatorname{Re} \left( \frac{1}{2} s_{ABC} z^B z^C \right) \right), \\ h_0 &= 2e^{K/2} (\cos \alpha \operatorname{Im}(z^1 z^2 z^3) - \sin \alpha \operatorname{Re}(z^1 z^2 z^3)), \end{aligned} \quad (\text{A.3})$$

where  $K$  is defined by (2.3). It is understood that in (A.3) all moduli dependence is evaluated at spatial infinity, *i.e.*  $z^A = z_\infty^A$ .

Restricting the discussion to Type IIA compactified on a 6-torus (in its STU-truncation), the factor  $\Sigma$  in (A.1) is uniquely given by

$$\begin{aligned} \Sigma^2(H) &= - (H_\Lambda H^\Lambda)^2 + 4 (H^1 H_1 H^2 H_2 + H^1 H_1 H^3 H_3 + H^2 H_2 H^3 H_3) \\ &\quad - 4 H^0 H_1 H_2 H_3 - 4 H_0 H^1 H^2 H^3. \end{aligned} \quad (\text{A.4})$$

Notice  $\Sigma^2(H)$  is nothing but the quartic invariant (2.10) in which all charges  $\Gamma = (p^\Lambda; q_\Lambda)$  have been replaced by the harmonic functions  $H = (H^\Lambda; H_\Lambda)$ .

The off diagonal metric components can be found explicitly by solving

$$\star d\omega = \langle dH, H \rangle, \quad (\text{A.5})$$

where  $\star$  is the Hodge dual on flat  $\mathbb{R}^3$ . The Dirac parts  $\mathcal{A}_d^A$ ,  $\omega_0$  of the vector potentials can be obtained from

$$d\omega_0 = \star dH^0, \quad d\mathcal{A}_d^A = \star dH^A. \quad (\text{A.6})$$

Regularity of the solution requires  $N - 1$  independent consistency conditions on the relative positions of the  $N$  centers, reflecting the fact that these configurations really are interacting and one can't move the centers around freely. These conditions arise from requiring integrability of (A.5)

$$\langle H, \Gamma_i \rangle|_{x=x_i} = 0, \quad (\text{A.7})$$

or written out more explicitly

$$\sum_{b \neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} = \langle h, \Gamma_a \rangle, \quad \text{with} \quad \langle \Gamma_i, \Gamma_j \rangle = -p_i^0 q_0^j + p_i^A q_A^j - q_A^i p_j^A + q_0^i p_j^0. \quad (\text{A.8})$$

where  $r_{ab} = |x_a - x_b|$ . Consequently, the equilibrium distances between the different centers depend on the asymptotic values of the scalar fields and on the charges at each center.

A crucial property of these multi-centered solutions is that they carry intrinsic angular momentum due to rotations on  $\mathbb{R}^3$ , which equals to

$$\vec{J} = \sum_{i < j} \frac{1}{2} \langle \Gamma_i, \Gamma_j \rangle \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}. \quad (\text{A.9})$$

Due to the off-diagonal terms in the metric sourcing this angular momentum, there are further requirements this set of configurations have to satisfy to prevent the existence of closed timelike curves (CTC). These are guaranteed to be absent if

$$\Sigma^2 > \omega_i \omega^i, \quad (\text{A.10})$$

a condition that has to be satisfied everywhere, and not just point wise [48, 49].<sup>13</sup>

Assuming a given charge vector  $\Gamma_i$  solves the attractor equations, the behavior of the multi-centered solution close to the center  $\vec{x}_i$  is fully determined by the charges in  $\Gamma_i$ , due to the attractor mechanism. In particular,  $\Sigma$  is a function of the entropy of the pole, *i.e.* the quartic invariant evaluated at that center. But depending on the amount of supersymmetry preserved by the state associated with  $\Gamma_i$ , such entropy might vanish. Under these circumstances, the order of the pole changes. More importantly, the value of the pole will *no* longer be determined by the attractor mechanism. For now, we are just interested in matching the order of the pole with the amount of supersymmetry preserved by the state.

According to the discussion in [35], the prescription is that by looking at the scaling of  $\Sigma^2(H)$  with respect to the distance to the center  $\rho = |x - x_i| \rightarrow 0$ , one finds :

$$\begin{aligned} 1/8 \text{ BPS} , & \quad \Delta > 0 , & \quad \Sigma^2 \propto \rho^{-4} \\ 1/8 \text{ BPS} , & \quad \Delta = 0 , & \quad \Sigma^2 \propto \rho^{-3} \\ 1/4 \text{ BPS} , & \quad \Delta = 0 , \partial\Delta = 0 , & \quad \Sigma^2 \propto \rho^{-2} \\ 1/2 \text{ BPS} , & \quad \Delta = 0 , \partial\Delta = 0 , \partial^2|_{Adj}\Delta = 0 , & \quad \Sigma^2 \propto \rho^{-1} \end{aligned} \quad (\text{A.11})$$

where the symbol  $\partial$  denotes derivatives with respect to the charges  $p^\Lambda$  and  $q_\Lambda$ .

## B Algebraic description of 1/4 and 1/2 BPS states

Both 1/4 and 1/2 BPS states have vanishing quartic invariant and vanishing  $\partial\Delta/\partial q_\Lambda = \partial\Delta/\partial p^\Lambda = 0$ . The latter set of conditions is :

$$\frac{\partial\Delta}{\partial q_0} = 2p^0(p^\Lambda q_\Lambda) - 4p^1 p^2 p^3 = 0, \quad (\text{B.1})$$

$$\frac{\partial\Delta}{\partial p^0} = 2q_0(p^\Lambda q_\Lambda) - 4q_1 q_2 q_3 = 0, \quad (\text{B.2})$$

$$\frac{\partial\Delta}{\partial q_A} = -2p^A(p^\Lambda q_\Lambda) + 4p^A \sum_{B \neq A} p^B q_B - 2p^0 s_{ABC} q_B q_C = 0, \quad (\text{B.3})$$

$$\frac{\partial\Delta}{\partial p^A} = -2q_A(p^\Lambda q_\Lambda) + 4q_A \sum_{B \neq A} p^A q_B - 2q^0 s_{ABC} p^B p^C = 0. \quad (\text{B.4})$$

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<sup>13</sup>This condition may not be satisfied and the configuration still be free of these causal pathologies, *i.e.* this condition is sufficient, but not necessary.

Let us assume  $p^0, q_0 \neq 0$ . Using (B.1) and (B.2), we learn that

$$p^0 q_1 q_2 q_3 = q_0 p^1 p^2 p^3 \quad \Leftrightarrow \quad p^\Lambda q_\Lambda = \frac{2}{q_0} q_1 q_2 q_3 = \frac{2}{p^0} p^1 p^2 p^3.$$

Multiplying (B.3) with  $q_A$  (without summing over the index  $A$ ) we obtain :

$$-2p^A q_A (p^\Lambda q_\Lambda) - 4p^0 q_1 q_2 q_3 + 4p^A q_A \sum_{B \neq A} p^B q_B = 0. \quad (\text{B.5})$$

Using the identities :

$$\begin{aligned} 4p^A q_A \sum_{B \neq A} p^B q_B &= -4(p^A q_A)^2 + 4p^A q_A p^0 q_0 + 4p^A q_A (p^\Lambda q_\Lambda), \\ -2p^A q_A (p^\Lambda q_\Lambda) - 4p^0 q_1 q_2 q_3 &= -2(p^\Lambda q_\Lambda) (p^0 q_0 + p^A q_A), \end{aligned}$$

we can write (B.5) as

$$2((p^\Lambda q_\Lambda) - 2p^A q_A) (p^A q_A - p^0 q_0) = 0,$$

where there is still no summation over the index  $A$ . It is convenient to introduce the auxiliary variables  $x_0 = p^0 q_0$  and  $x_A = p^A q_A$  for  $A = 1, 2, 3$  to solve this equation :

$$(-x_0 + x_1 + x_2 + x_3 - 2x_A) (x_A - x_0) = 0.$$

In terms of these variables, it is easy to find the general solution :

$$x_A = x_0, \quad x_B = x_C \quad A \neq B \neq C$$

up to permutations in the three tori, i.e.  $A \leftrightarrow B \leftrightarrow C$ . It is the above fact that allows us to write the charge vector in terms of eight parameters  $\{\alpha_{1,2}, \beta_{1,2}\}$  and  $\{P^0, P^2, P^3, Q_0\}$  :

$$\Gamma = (\beta_1 P^0, \beta_2 P^0, \alpha_1 P^2, \alpha_1 P^3, \beta_1 Q_0, \alpha_2 P^3, \alpha_2 P^2, \beta_2 Q_0) \quad (\text{B.6})$$

Inserting this expression in our initial set of equations (B.1)-(B.4), we obtain :

$$\begin{aligned} P^0 P^2 P^3 \alpha_{1,2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) &= 0, \\ Q_0 P^2 P^3 \alpha_{1,2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) &= 0, \\ Q_0 P^0 P^{2,3} \beta_{1,2} (\beta_1 \alpha_2 - \beta_2 \alpha_1) &= 0. \end{aligned} \quad (\text{B.7})$$

whereas the vanishing of the quartic invariant  $\Delta$  requires :

$$\Delta = -4 (\beta_1 \alpha_2 - \beta_2 \alpha_1)^2 P^0 P^2 P^3 Q_0.$$

This latter constraint is not independent, since whenever all the first derivatives of  $\Delta$  vanish, the quartic invariant itself also does.

Thus, for non-vanishing parameters, the solution will always be given by  $\beta_1 \alpha_2 = \beta_2 \alpha_1$ . But we can still satisfy (B.7) by setting a combination of coefficients  $(\alpha_{1,2}, \beta_{1,2})$  and/or charges  $(P^0, P^2, P^3, Q_0)$  to vanish.

The previous derivation assumed that both  $(p^0, q_0)$  were not vanishing.<sup>14</sup> It is easy to extend the analysis when either of them vanishes.

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<sup>14</sup>Strictly speaking, when multiplying our initial equations by  $q_A$  and  $p^A$  we were also assuming all charges were generically non-vanishing.

**The  $q_0 = 0$  branch.** Equation (B.2) implies the product  $q_1 q_2 q_3$  vanishes. Let us pick one of them to vanish, i.e.  $q_A = 0$  (for some  $A=1,2,3$ ), having  $q_B, q_C \neq 0$  for  $B \neq C \neq A$ . In this situation,  $(p^\Lambda q_\Lambda) = x_B + x_C$ , where  $x_B$ 's were defined as above. The non-trivial equations to solve become :

$$\begin{aligned}\frac{\partial \Delta}{\partial q_B} &= 2p^B (x_C - x_B) = 0, \quad B \neq C, B, C \neq A \\ \frac{\partial \Delta}{\partial p^B} &= 2q_B (x_C - x_B) = 0, \quad B \neq C, B, C \neq A \\ \frac{\partial \Delta}{\partial q_A} &= 2p^A (p^\Lambda q_\Lambda) - 4p^0 q_C q_B = 0, \\ \frac{\partial \Delta}{\partial q_0} &= 4p^1 p^2 p^3 - 2p^0 (p^\Lambda q_\Lambda) = 0.\end{aligned}$$

If all charges appearing above are generically non-zero, the solution is given by :

$$x_B = x_C \quad \text{and} \quad p^0 = \frac{p^A p^B}{q_C},$$

which is the particular case  $Q_0 = 0$  in the charge vector (B.6).

If we do not impose  $x_B = x_C$ , we are forced to allow charges to vanish, and we always end up satisfying  $x_B = x_C = 0$ . The most general set of solutions in this category are summarised by

$$(p^0, [p^A, p^B]; [q_C], 0) \quad \text{and} \quad (0, [p^A]; [q_B, q_C], 0)$$

which do still belong to the class described by (B.6), without necessarily satisfying the condition  $\beta_1 \alpha_2 = \beta_2 \alpha_1$ .

**The  $p^0 = 0$  branch.** The analysis of this branch is completely analogous to the one above. In this case, one of the  $p^A$  charges has to vanish because of (B.1). If all remaining charges are non-vanishing, we again have  $x_C = x_B$ , with  $q_0 = q_A q_B / p^C$ . If extra charges are allowed to vanish, all solutions are included in either of the following two sets :

$$(0, [p^A, p^B]; [q_C], 0) \quad \text{and} \quad (0, [p^A]; [q_B, q_C], q_0)$$

which do still belong to the class described by (B.6), without necessarily satisfying the condition  $\beta_1 \alpha_2 = \beta_2 \alpha_1$ .

**Conclusion.** The analysis presented above proves that any 1/4 or 1/2 BPS state has a charge vector of the form (B.6) :

$$\Gamma = (\beta_1 P^0, \beta_2 P^0, \alpha_1 P^2, \alpha_1 P^3, \beta_1 Q_0, \alpha_2 P^3, \alpha_2 P^2, \beta_2 Q_0)$$

where either  $\beta_1 \alpha_2 = \beta_2 \alpha_1$ , or whenever  $\beta_1 \alpha_2 \neq \beta_2 \alpha_1$ , there are enough vanishing coefficients and/or charges so that (B.7) are still satisfied.



## C Moduli identities

In this appendix we gather some useful expression relating constant asymptotic value of the harmonic functions  $(h^\Lambda, h_\Lambda)$  and the moduli  $z^A$  that we used in section 4. The total charge of the system is  $\Gamma_{D0-D6}$  and the central charge is

$$Z_{D0-D6} = e^{K/2}(p^0 z^1 z^2 z^3 - q_0) = |Z_{D0-D6}| e^{i\alpha} \quad (C.1)$$

Starting from the definitions (A.3), we have

$$\begin{aligned} h^1 &= -\frac{2e^{K/2}}{|Z_{D0-D6}|} (q_0 J^1 + p^0 |z^1|^2 \text{Im}(z^2 z^3)) , \\ h_1 &= -\frac{2e^{K/2}}{|Z_{D0-D6}|} (q_0 \text{Im}(z^2 z^3) + p^0 J^1 |z^2 z^3|^2) , \end{aligned} \quad (C.2)$$

and similarly expressions for  $h^{2,3}$  and  $h_{2,3}$ , where  $z^A = B^A + iJ^A$ . In the function  $\Sigma^2$  for type **I** bound states the following combinations appear

$$\begin{aligned} h_2 h_3 - h^1 h_0 &= 4e^K |z^1|^2 J^2 J^3 , \\ h^2 h^3 - h_1 h^0 &= 4e^K J^2 J^3 , \\ h_2 h^2 + h_3 h^3 - h_1 h^1 - h_0 h^0 &= 8e^K B^1 J^2 J^3 . \end{aligned} \quad (C.3)$$

Linear combinations of these terms in (4.15) simplify to  $(4e^K J^2 J^3 |az^1 - c|^2)$  and  $(4e^K J^2 J^3 |dz^1 - b|^2)$ . Finally, other useful identities are

$$\begin{aligned} h^1 h_1 - h_0 h^0 &= 4e^K J^1 \text{Im}(z^2 z^3) , \\ h^2 h_2 - h_0 h^0 &= 4e^K J^2 \text{Im}(z^1 z^3) , \\ h^3 h_3 - h_0 h^0 &= 4e^K J^3 \text{Im}(z^1 z^2) . \end{aligned} \quad (C.4)$$

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